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DISCUSSION OF END RESTRAINTS ON TRUSS MEMBERS

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By Jack R. Benjamin, George Winter, Abraham Slavin, J. Edmund Fitzgerald, Joseph S. Newell, Charles W. Dohn, and Harold E. Wessman and Thomas C. Kavanagh

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DISCUSSION

Jack R. Benjamin,[19] Assoc. M. ASCE.—For their clear presentation of an exceedingly complex subject the authors are to be complimented. Although ultimate failure by instability is the criterion stipulated in the paper, applications to cases where unit stress controls, rather than instability, are important. The unit stress limit merely provides an alternate failure criterion.

The variation in stiffness and carry-over factor for columns under loading as contrasted to beams under loading is important in the design of building frames. In building frames the column loads are almost independent of girder moments so that a single adjustment in stiffness and carry-over factor is sufficient in an elastic analysis.

Two factors are neglected by the authors—influence of secondary stress moments and moments caused by initial curvature. Secondary stress influences will affect the buckling load of the entire truss. In addition, although the initial curvature of truss members does not influence stiffness or carry-over factors, it does add initial joint moments to the moment distribution procedure. These moments are approximately proportional to the axial loads and can be of the order of magnitude of 50% of the allowable moment in the member when initial curvatures specified by the current tolerances of the American Institute of Steel Construction are used. In an elastic analysis, these curvature moments can be significant. Whether or not these two influences reduce the buckling load of a truss a significant amount deserves investigation.

George Winter,[20] M. ASCE.—Methods of investigating the buckling stability of structural frameworks are basically of two types: (*a*) "Classical methods" of solving the simultaneous transcendental equations which determine the buckling load; and (*b*) numerical methods of successive approximation. A very interesting historical review of the former was reported by Mr. Kavanagh[18] in 1948. For customary trusses with a considerable number of bars the classical methods are too cumbersome mathematically. The two numerical methods most promising for practical use are: (1) The Lundquist-Hoff[4,11,12]

Note.—This paper by Harold E. Wessman and Thomas C. Kavanagh was published in September, 1949, *Proceedings*. The numbering of footnotes, equations, tables, and illustrations in this Separate is a continuation of the consecutive numbering used in the original paper.

[19] Asst. Prof. of Civ. Eng., Stanford Univ., Stanford University, Calif.

[20] Prof. and Head, Dept. of Structural Eng., Cornell Univ., Ithaca, N. Y

[18] "Instability of Plane Truss Frameworks," by T. C. Kavanagh, thesis presented to New York Univ., New York, N. Y., in April, 1948, in partial fulfilment of the requirements for the degree of Doctor of Engineering Science.

[4] "Stability of Structural Members Under Axial Load," by Eugene E. Lundquist, *Technical Note No. 617*, National Advisory Committee for Aeronautics, Washington, D. C., October, 1937.

[11] "Stable and Unstable Equilibrium of Plane Frameworks," by N. J. Hoff, *Journal of the Aeronautical Sciences*, January, 1941, p. 115.

[12] "Stress Analysis of Aircraft Frameworks," by N. J. Hoff, *Journal*, Royal Aeronautical Soc., July 1941, p. 241.

adaptation of the moment distribution approach, on which the present paper is based, and (2) the end restraint method developed by P. T. Hsu.[17] Method (2) has been published as part of an extensive study on buckling of trusses and rigid frames by the writer and his collaborators.[21] The contribution of the present paper consists in showing that the same end restraints which are found by method (2) can also be calculated from method (1). With regard to the implications of these various approaches the following comments can be made:

(1) In the Lundquist-Hoff method, ably restated by the authors, the stability of the truss is investigated at successively increasing loads until a load is reached for which the truss proves unstable. This procedure brackets the unknown exact critical load between the highest investigated load which proved stable and that load which showed instability. This bracketed range can be narrowed only by further time-consuming trials. The disadvantage of this approach is that each successive trial gives merely a new lower limit for the critical load so that the next trial load must be guessed, and may prove to be too high or too low by a considerable amount. (There are ways to reduce the required number of trials in the Lundquist-Hoff method; since no mention of them was made by the authors, they will not be discussed here.) In connection with this method the determination of restraints and effective lengths is wholly superfluous, as, in fact, is stated by the authors. After showing how to compute these quantities, the authors make no practical use of either of them.

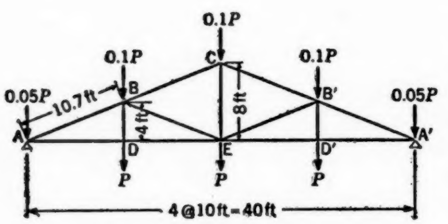

FIG. 7.—ILLUSTRATIVE EXAMPLE

In contrast, the end restraint approach[17,21] gives a method of determining restraints and effective lengths directly without a detour through moment distribution. These quantities are then used to calculate critical loads directly. Here, too, the truss is investigated under successively increasing loads until the critical load is bracketed sufficiently closely. However, the advantage of this method is that each successive step gives both an upper limit and a lower limit for the critical load which reduces considerably the required number of trials. This fact is easily illustrated by the example of the roof truss with heavy, suspended ceiling analyzed in detail by the writer and his collaborators,[22] as shown in Fig. 7.

For a trial load $P = 38$ kips, using end restraints, pertinent information was obtained for the three compression members, as listed in Table 4, in which k is the effective length coefficient; N_{38} is the compression force in a member

[17] "Elastic Stability of Members in Trusses," by P. T. Hsu, thesis presented to Cornell Univ., Ithaca, N. Y., in May, 1947, in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

[21] "Buckling of Trusses and Rigid Frames," by George Winter, P. T. Hsu, B. Koo, and M. H. Loh *Bulletin No. 36*, Cornell Univ. Eng. Experiment Station, Ithaca, N. Y., April, 1948.

[22] *Ibid.*, pp. 35–40.

caused by the loading $P = 38$ kips; N_{cr} is the critical load of the individual member for restraints determined for $P = 38$ kips; and $P_{N(cr)}$ is the loading P that would produce a bar force N_{cr} (for example, to have 177.9 kips in bar AB, a loading of $P = 40$ kips must be applied). Two facts are demonstrated by Table 4:

TABLE 4.—ROOF TRUSS ANALYSIS

Member	k	N_{38}	N_{cr}	$P_{N(cr)}$
AB.......	0.720	169.0	177.9	40.0
BC.......	0.885	112.2	117.8	39.8
BE.......	0.685	55.2	74.1	51.0

(*a*) The truss is stable at $P = 38$ kips since the critical bar forces are larger than those caused by this loading or, in other words, all $P_{N(cr)}$-values are larger than 38 kips. The latter value, therefore, is a lower limit for the exact critical load.

(*b*) On the other hand, the smallest of the $P_{N(cr)}$-values—that is, 39.8 kips—is an upper limit for the true critical load because, with increasing load, restraints decrease or (at best) remain practically constant so that, in turn, effective lengths increase, unless they also stay constant (see Fig. 6). Thus, with a load larger than 38 kips, say, 39 kips, all N_{cr}-values will be found smaller or, at best, equal to the tabulated values. Hence the corresponding $P_{N(cr)}$-values cannot exceed those found for 38 kips.

This particular trial, therefore, was found to bracket the exact critical load P between 38 kips and 39.8 kips, that is, within a 5% interval. For practical purposes, no further trial would be needed.

The reason for condition (*b*) is that with increasing axial load the effective rigidities of the compression members decrease. On the other hand, the rigidities of the tension members increase, but at a much slower rate and with generally negligible effect. Consequently, with increasing load on the truss the restraints provided by abutting members to the compression members generally decrease and result in a decreasing or, at best, a constant N_{cr}.

(2) Another substantial advantage of the restraint method, not mentioned by the authors, refers to the problem of design rather than that of analysis. To be sure, in a mathematical sense trusses fail as units by "one hoss shay" action. The authors state, correctly, that despite this action some "weaker" members of the truss contribute more to limiting its critical load than do other, more husky members. To illustrate: If, in the foregoing truss example, the critical load of slightly more than 38 kips was found insufficient for the given, actual design loads, the truss would have to be redesigned and strengthened. Theoretically the strengthening of any member will strengthen the entire truss. However, it is more effective to strengthen some members than others and the designer will want to change these members for maximum economy. The tabulated values show immediately that for this particular truss the members AB and BC should be strengthened, whereas substituting a heavier member for BE would have a small effect on the carrying capacity. This fact is evident because, for members AB and BC, the N_{cr}-values exceed the actual

bar forces N_{38} only slightly, whereas the difference between these values exceeds 30% for member BE. The latter, therefore, has excess strength which is drawn upon by members AB and BC to supplement their own insufficient rigidity. The determination of such "weakest members" rather than the determination of the "weakest joints" discussed by the authors is important since it is not a joint but a member that must be redesigned for the desired effect. The determination of these "weakest members" is an automatic feature of the end restraint method and requires no additional computation.

(3) In some structures, particularly those subject to fatigue, the determination of actual maximum stresses at loads below the critical is important, and governs design. These stresses will exceed the value of P/A if lateral loads, eccentricities, or other imperfections induce bending. Such calculations are greatly facilitated by determination of end restraints for these subcritical loads. Corresponding methods (including tables and charts) were reported in 1948[21] as a further development of the end restraint method. A chart of the general type of Fig. 4[8] is also essential in connection with end restraint methods. However, negative as well as positive restraints can, and do, occur; and the chart developed at Cornell University, in Ithaca, N. Y., includes the range of such negative restraints, in contrast to Fig. 4.

(4) As stated by the authors, the methods in the paper apply only to frames without joint translation. The problem of effective length or of actual critical load, however, is of particular importance for rigid frames not restrained against sidesway. Fig. 8[21] shows the tremendous influence of sidesway restraint on the critical load of simple portal frames, for which

$$P_{cr} = \frac{\pi^2 E I}{(k\,l)^2} \qquad (24)$$

If such restraint is provided, effective length coefficients do not exceed 0.7 and 1.0 for the cases of fixed base or hinged base, respectively; without such restraint the corresponding maxima are 2.0 and ∞, respectively. The upper limit of $k = 2$ advanced by the authors in their last paragraph is seen to be correct, therefore, only for frames with completely fixed bases. It will be exceeded considerably for frames hinged or only partly restrained at the footings. The latter condition is true of most soils. Since buckling loads are inversely proportional to k^2, consideration of sidesway buckling is of paramount importance for rigid frames.

(5) In one of their concluding statements the authors seem to indicate that effective length determinations are superfluous for ordinary steel trusses and that compression members of such trusses can be designed on the basis of yield strength without regard to buckling. Some skepticism concerning this contention seems justified. As an example, at the critical load, values of P/A for the roof truss in Fig. 7—with slenderness ratios varying from 145 to 175 (that is, within the range allowed by current codes)—range from approximately 16

[8] "Elastically Encastred Struts," by N. J. Hoff, *Journal*, Royal Aeronautical Soc., September, 1936, p. 663.

kips per sq in. to 25 kips per sq in. This range is considerably lower than the yield point of 33 kips per sq in. for mild steel, not to mention the still higher yield points that apply to the alloy steels mentioned by the authors. To be sure, current truss practice usually results in smaller slenderness values than those of this example; and, therefore, the design approaches more closely the situation depicted by the authors—probably due to the overly conservative column requirements in codes, particularly in older codes during whose lifetime present design habits have developed. It would seem that investigations of

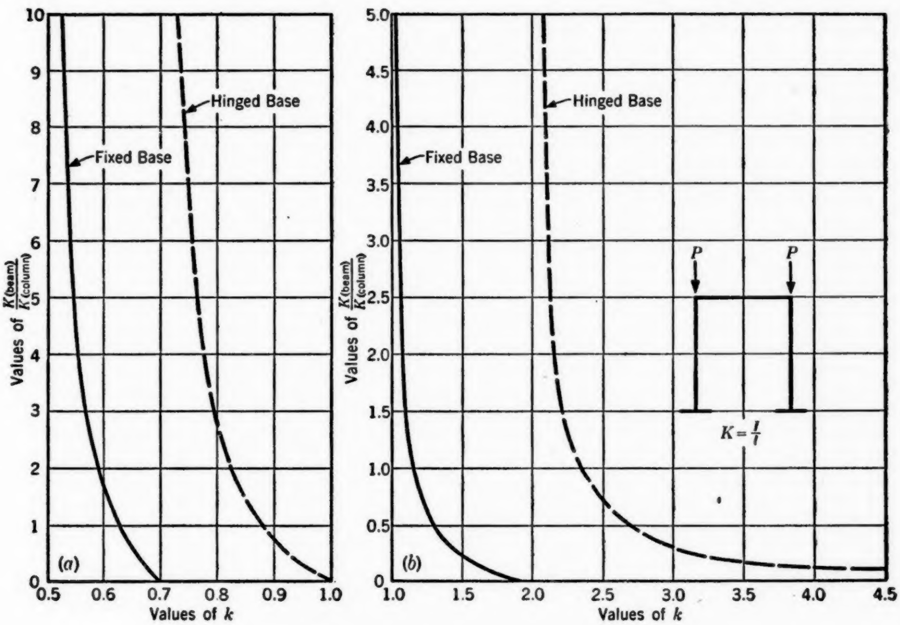

FIG. 8.—EFFECTIVE LENGTH COEFFICIENTS k FOR PORTAL FRAMES
(a) Portal Frame, Sidesway Prevented
(b) Portal Frame, Sidesway Permitted

frame buckling such as those discussed are justified only if they promote the establishment of more realistic and, most likely, more liberal column design procedures. Once such procedures are introduced, a much larger number of structures will have proportions which require reasonably accurate determinations of effective length. Most design codes in the United States contain no reference to effective length variation. In contrast, the British code[23] dated 1948 makes explicit allowance for this factor, fifteen pages of that code being devoted exclusively to stipulations regarding effective lengths.

[23] "The Structural Use of Steel in Buildings," *British Standard Code of Practice, CP 113*, London, 1948.

ABRAHAM SLAVIN,[24] M. ASCE.—Average values of length reduction factors for compression members in civil engineering trusses are given by Friedrich Bleich,[25] M. ASCE, as $k = 0.80$. The Second Progress Report of the Special ASCE Committee on Steel Column Research[26] notes that from tests on pin-ended columns the suggested value of k is 0.78. E. H. Salmon[27] mentions that the k-value for practical end conditions probably lies between 0.56 and 1.00 and suggests the value of 0.78. For light roof trusses having members attached by small gusset plates to adjacent members of less stiffness, he proposes the value of $k = 1.0$. The late Leon S. Moisseiff,[28] M. ASCE, gave, as the nearest simple equivalent to an actual compression member in civil engineering structures, a column with partly restrained ends for which $k = 0.75$, or the average of the theoretical value 0.50 for fixed ends and the theoretical value 1.0 for pinned ends in an ideal column. A. S. Niles, Assoc. M. ASCE, advises that for airplane trusses of welded jointed tubular sections, designed by ultimate load formulas, the value of $k = 0.70$ in the prime compression member (highest value of L/j at critical load) was practically substantiated by static tests to failure made at an army airfield in United States during 1920 and later. This value is practically equal to that theoretically derived for a compression member having one end restrained and the other end pinned, although both ends of the truss column were actually subject to an intermediate degree of restraint. J. S. Newell reports that, for most of the airplane truss frames he analyzed, the critical load corresponded with the prime compression member having a k-value of about 0.70.

The foregoing results are only a few of those from numerous reports regarding restraint coefficients. Perhaps the difficulty, if any, is that many of the published column formulas do not indicate the type of end restraint, and it is assumed that the column formulas are for the practical compression members which may have varying degrees of end restraint. In fact, the column formula specified by the American Institute of Steel Construction does not indicate the type of end restraints; and, from discussion with well-known structural research engineers, the general assumption is made that the formula applies to a practical column with no definite or theoretical value for the end restraints.

In the fourth paragraph of their paper, the authors state:

> "It is not generally recognized that, as loads are increased on a framework, the end restraints acting on a compression member in the framework do not remain constant."

From the experience of the writer he believes that these facts are fairly widely recognized. Computed restraint factors from the stability studies of the plane

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[24] Cons. Civ. Engr. and Architect, New York, N. Y.; and with Dept. of Civ. Eng., Polytechnic Inst. of Brooklyn, Brooklyn, N. Y.

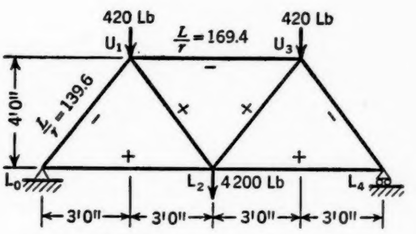
[25] "Die Knickfestigkeit Elasticher Stabverbindungen," by Friedrich Bleich, *Der Eisenbau*, April, 1919, p. 83.

[26] "Steel Column Research: Second Progress Report of the Special Committee," *Transactions*, ASCE, Vol. 95, 1931, p. 1201.

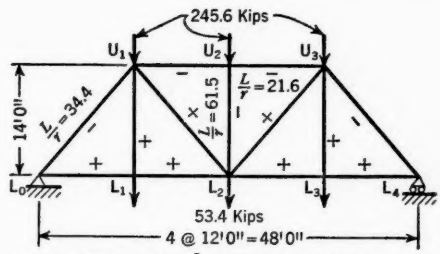
[27] "Materials and Structures," by E. H. Salmon, Longmans, Green & Co., London and New York, Vol. 2, 1937.

[28] "Design Specifications for Bridges and Structures of Aluminum Alloy 27 S-T," by Leon S. Moisseiff Aluminum Co. of America, Pittsburgh, Pa., Revised Ed., March, 1940.

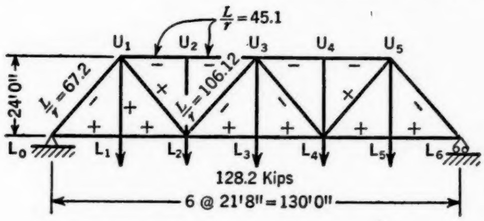
TABLE 5.—Critical Values for Trusses [a]



(a) AIRPLANE TRUSS



(b) BUILDING TRUSS, ($\frac{L}{r}$ Values for Study No. 1)



(c) HIGHWAY TRUSS

Study	Design formula	Yield point[b]	$\overline{E}$	Critical load factor	Critical Values for Member: L_0U_1 Stress[b]	L_0U_1 k	U_1U_2[c] Stress[b]	U_1U_2[c] k
				(a) Airplane Truss				
1	Ultimate	36,000	E_u	1.0310	24,307	0.715	19,320[d]	0.704[d]
2	Ultimate	36,000	E	1.0452	24,648	0.758	19,589[d]	0.701[d]
				(b) Building Truss				
1	Old AREA[e]	36,000	E_u	2.5553	35,035	0.834	35,681[d]	0.765[d]
2	Old AREA[e]	40,000	E_u	2.8372	38,900	0.829	39,617[d]	0.781[d]
3	AISC[f]	36,000	E_u	2.1509	34,969[d]	0.766[d]	34,294	1.532
4	AISC[f]	40,000	E_u	2.3868	38,805[d]	0.768[d]	38,055	1.524
5	Old AREA[e]	36,000	E_T	2.57791	35.343	1.530	35,998[d]	0.572[d]
6	AISC[f]	36,000	E_T	2.21109	35,948[d]	0.715[d]	35,254	2.185
7	AISC[f]	33,000	E_T	2.0272	32,960[d]	0.704[d]	32,321	2.228
8	AISC[f]	36,000	E_u	2.1344	34,701[d]	0.860[d]	35,072	1.064
9	AISC[f]	40,000	E_u	2.3680	38,499[d]	0.861[d]	38,911	1.073
				(c) Highway Truss				
1	AREA (1943)[e]	36,000	E_u	2.3983	32,209	0.847	34,536[d]	0.786[d]
2	AREA (1943)[e]	33,000	E_T	2.2865	30,708	1.150	32,926[d]	0.703[d]

[a] All Warren type trusses except studies 8 and 9, Table 5(b), which are Howe type trusses. [b] Pounds per square inch. [c] In the airplane truss, Table 5(a), this becomes member U_1U_3. [d] Values for the member with the highest ratio L/j. [e] Old specifications of the American Railway Engineering Association, as distinguished from the 1943 specifications. [f] American Institute of Steel Construction.

trusses of three types[14] are given in Table 5. The k-values were obtained by the simple method of applying the Euler formula with the reduced modulus.

The airplane truss in Table 5, designed by ultimate load formulas, is the same as that used by the authors (except for engineering notation) and by Messrs. Niles and Newell.[29] The building truss by E. J. Squire, M. ASCE, and the highway truss,[14] both designed by working load formulas, are included as technical supporting data for values of k, and as an illustration of the relation of critical stresses to yield point values. For the Howe type of building truss (studies 8 and 9, Table 5(b)), the interior web members are reversed in direction to parallel the end posts. In this discussion, E_u refers to the effective modulus evaluated from the basic column formula; E_T is the tangent modulus evaluated from a typical stress-strain curve for structural steel; E_D is the double modulus; and E is the elastic modulus. In the case of the airplane truss, E may be used instead of E_u with an increase in value of the critical load of only 1.5%.

For both studies of the airplane truss, k may be taken as practically equal to 0.7. The first four studies of the building truss, and the first study of the highway truss, on the basis of the conservative modulus E_u, indicate a k-value close to 0.78, which has been suggested instead of unity for pin-ended columns in civil engineering structures to allow for the friction at the pin joints. For building trusses of the Howe type (studies 8 and 9), on the basis of E_u, $k = 0.86$. For the studies using E_T, with the exception of study 5, the k-values are about 0.7, or the same as those evaluated for the airplane truss with the modulus concept of E_u. The restraint coefficients will vary with loading, but the value at the critical load is of interest. All the data in Table 5 were evaluated and compiled prior to, and independently of, the preparation of the paper by the authors and were reported to them as part of the group research work.

In presenting Eq. 2, the authors mention that the tangent modulus is now generally accepted by aeronautical engineers and that it has also been accepted by the Column Research Council of the Engineering Foundation. However, in the computations for the airplane truss, the authors (in applying Eq. 15), as well as the writer, used the reduced modulus E_u. Since the effective modulus value is important for computing the critical loads and the corresponding restraint coefficients for the compression members in the framework, comment regarding the evaluation of the reduced modulus is warranted. The value of E_u is lower than the value of the tangent modulus and is intended to replace the Engesser and the von Kármán formulas that make no allowance for the effect of the practical column as compared to that of the corresponding ideal column. With a low value of the reduced modulus, the numerical value of L/j is increased and the computed theoretical critical load is a conservative value. This method of computing the reduced modulus is given by Messrs. Niles and Newell.[30] The report of the Army-Navy-Civil Committee on Aircraft Design Criteria[31]

[14] "Stability Studies of Structural Frames," by A. Slavin, thesis presented to New York Univ., New York, N. Y., in May, 1948, in partial fulfilment of the requirements for the degree of Doctor of Engineering Science.

[29] "Airplane Structures," by A. S. Niles and J. S. Newell, John Wiley & Sons, Inc., New York, N. Y., 3d Ed., Vol. II, 1943, p. 306.

[30] *Ibid.*, Vol. I, 1943, pp. 332–334.

[31] "Strength of Aircraft Elements," *ANC-5*, Army-Navy-Civil Committee on Aircraft Design Criteria, Revised Ed., Washington, D. C., December, 1942 (as amended August, 1946).

also states that the modified Euler formula does not have much practical importance in determining the short column curve, but that it is of practical interest in connection with the determination of the effective modulus that can be used to compute instability stresses. N. J. Hoff[32] mentions that the specification of a short column formula establishes a uniquely determined connection between the reduced modulus and the elastic modulus. W. Prager,[16] Eugene E. Lundquist and Claude M. Fligg,[33] and Mr. Moisseiff and Frederick Lienhard,[34] M. ASCE, also have used E_u.

Recommendations for the tangent modulus by F. R. Shanley[3] and O. H. Basquin[35] are based on the ideal unit columns. The recommendations by Mr. Shanley are essentially a verification of the results of prior laboratory work on ideal columns by the Aluminum Research Laboratories[36,37] at New Kensington, Pa. K. Borkmann[38] suggests an expedient estimate of the effective modulus on the basis of a linear relation between the proportional limit and the yield point. The writer[14] proposes an effective modulus based on a linear relation between seven tenths of the yield point and the yield point, which is intended for structural steels. If seven tenths of the yield point value is above the known value of the proportional limit, it does not apply and in its place the Borkmann modulus is suggested. It is the considered judgment of the writer that allowance should be made in the modulus value for the difference between the practical column and the ideal column.

In the derivation of their formulas, the authors use a spring constant s, and M is the applied moment equal to S, the stiffness for the far end fixed. Since a unit moment is applied for each loading state to determine the unbalanced moment or stability factor r, the assumption that $M = S$ is a limited condition. It is evident that with increased loading the stiffness of the members will decrease, although the same unit moment is applied in the computation procedure. The writer's computations[14] indicate that, for the airplane, building, and highway trusses (Table 5), with the far ends fixed (as is commonly assumed in the analysis of the entire framework), ΣS is positive for the loading above the critical determined by the stability $r = 1.0$. For the trusses in Table 5, the difference in critical load on the basis of the stiffness criterion $\Sigma S = 0$ and the series criterion $r = 1.0$ is about 17% for the airplane truss, about 0.4% for the building truss, and about 1.4% for the highway truss.

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[32] "A Note on Inelastic Buckling," by N. J. Hoff, *Journal of the Aeronautical Sciences*, April, 1944, p. 164.

[16] "The Buckling of an Elastically Encastred Strut," by W. Prager, *Journal*, Royal Aeronautical Soc., November, 1936, p. 833.

[33] "A Theory for Primary Failure of Straight Centrally Loaded Columns," by Eugene E. Lundquist and Claude N. Fligg, *Technical Report No. 582*, National Advisory Committee for Aeronautics, Washington, D. C., 1937.

[34] "Theory of Elastic Stability Applied to Structural Design," by Leon S. Moisseiff and Frederick Lienhard, *Transactions*, ASCE, Vol. 106, 1931, pp. 1052–1091.

[3] "Inelastic Column Theory," by F. R. Shanley, *Journal of the Aeronautical Sciences*, May, 1947, p. 261.

[35] "Tangent Modulus and the Strength of Steel Columns in Tests," by O. H. Basquin, *Journal of Research*, National Bureau of Standards, September, 1924, p. 381.

[36] "Column Strength of Various Aluminum Alloys," by R. L. Templin, R. G. Sturm, E. C. Hartmann, and M. Holt, *Technical Paper No. 1*, Aluminum Co. of America, Pittsburgh, Pa., 1938.

[37] "Typical Tensile and Compressive Stress-Strain Curves for Aluminum Alloy 24 S-T, Alclad 24 S-T, 24 S-RT, and Alclad 24 S-RT Products," by R. L. Templin, E. C. Hartmann, and D. A. Paul, *Technical Paper No. 6*, Aluminum Co. of America, Pittsburgh, Pa., 1942.

[38] "Charts for Checking the Stability of Compression Members in Trusses," by K. Borkmann, *Technical Memorandum No. 800*, National Advisory Committee for Aeronautics, Washington, D. C., July, 1936.

For the building truss in Table 5(*b*), the relative stiffness of the joints under increase in loading is shown in Fig. 9. Joint U_2 has the highest relative stiffness at the design load up to a load factor of about 2.52, beyond which, with load increase, it becomes the joint of least stiffness. Thus, the weakest joint is not the one having the least stiffness at the design load but rather the joint having the greatest rate of loss of stiffness with loading increase. The authors have also verified this statement with reference to the airplane truss. Since the joint stiffness is the sum of the stiffness of all members entering the truss

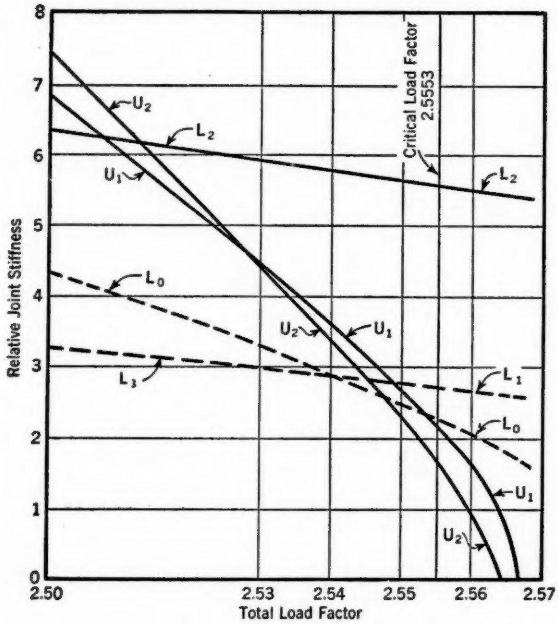

FIG. 9.—JOINT STIFFNESS; BUILDING TRUSS, STUDY 1

joint, the data in Fig. 9, which are related to the truss in Table 5(*b*), indicate that the stocky compression members at joint U_2, which have high unit stress, lose their contributing stiffness at the greatest rate because of the buckling tendency under increased loading; and, in turn, their respective restraint values are also reduced. The total stiffness of each joint has been found to decrease in relation to the prevalence of compression members composing the joint, and not in proportion to the increase in loading.

In the paragraph presenting Eq. 2, the authors claim that the buckling load in civil engineering trusses for a fixed-end column may be only slightly greater than that for the same column with pinned ends. The data in Table 6 from studies 1 of the trusses in Tables 5(*b*) and 5(*c*) cast some light on this question. The member having the highest ratio of L/j at the critical load, on the basis of E_u, is considered.

In stability computations, $L/j = \pi$ for pinned-end columns, and $L/j = 2\pi$ for theoretical fixed-end columns. A narrow range in load factors between these two conditions of end restraint is indicated for the cited typical civil engineering trusses. However, as noted by Mr. Hoff,[12] in actual frameworks the end fixity is elastic and is not perfectly rigid, and the upper limit of $L/j < 2\pi$ for a compression member with both ends rigidly fixed cannot be reached.

TABLE 6.—COMPARISON OF FIXED–ENDED COLUMNS AND PIN–ENDED COLUMNS FOR THE PRIMARY COMPRESSION TRUSS MEMBERS HAVING THE HIGHEST RATIOS L/j AT CRITICAL LOADS
(Yield Point 36 Kips per Sq In.)

(a) MEMBER U_1U_2, BUILDING TRUSS				(b) MEMBER U_1U_2, HIGHWAY TRUSS			
Load factor	Stress (lb per sq in.)	Ratio, L/j	Percentage range[a]	Load factor	Stress (lb per sq in.)	Ratio, L/j	Percentage range[a]
2.539	35,455	3.142	0.0	2.335	33,625	3.142	0.0
2.555[b]	35,680	4.108	+0.63	2.398[b]	34,536	3.977	+2.70
2.568	35,865	6.288	+1.14	2.460	35,425	6.372	+5.35

[a] Percentage range in load factor. [b] Critical load factor.

The first and last values of the data for each truss in Table 6 may be used to approximate the buckling load, from which k-values may be estimated, since there is a small range in load factors between the upper and lower values of L/j. The data in Table 6 also indicate that the critical compressive unit stresses are close to the yield point. This reasoning does not apply to the airplane truss in Table 5(a), in the same sense that it applies to the airplane truss in Fig. 5, because the compression members have high slenderness ratios and they fail at unit stresses considerably below the yield point.

In Fig. 6, the plot of r versus the load factor for the airplane truss applies to a unit moment at joint U_1. With application of a unit moment at joint L_0, the r-values are lower up to the equivalent critical load value at $r = 1.0$ and then are higher. For verification of the equivalence of the critical load, where $r = 1.0$, the unit moment should be applied separately to each joint at which compression members enter. It will be observed that the critical load, where $r = 1.0$, is the same regardless of the joint at which the unit moment is applied, as discussed by the authors in theorem 3. It is evident, however, that, for a truss frame in which all members are interdependent, at the critical load the frame fails as a unit regardless of the reserve strength in any members, since the continuity of the frame is destroyed.

The authors credit Mr. Lundquist[4] for the stability criteria and Mr. Hoff[11,12] for his contribution. The analysis method should be called the Lundquist-Hoff method. Although the work by Mr. Lundquist is shown as

applicable to member groups of a truss and although it may be applied to the entire truss frame, the procedure by Mr. Hoff is definitely applicable as a valid solution for the entire framework. In addition, credit is due to Messrs. Niles and Newell[29] for their moment distribution method which modifies the usual procedure. In the joint at which the unit moment is applied, the carry-over values are held against further distribution, which thus materially shortens the computations for the stability factor r.

The authors note that, for compression members in steel trusses, with the usual range of slenderness ratios, the buckling load practically coincides with the yield point of the steel regardless of the end restraints; and therefore any elaborate analysis of stability and end restraint is unwarranted for a steel truss. It is well known that steel (whether carbon steel, silicon steel, or nickel steel) has a well-defined yield point and that the elastic modulus E is the same for all regardless of the different values for proportional limit and yield point. However, the statement by the authors should be qualified to apply to civil engineering structures designed on the basis of working load formulas. It does not apply to steel trusses used in airplanes, which are designed by ultimate load formulas and have slender compression members that fail at stresses considerably below the yield point.

The data in Tables 5 and 6 indicate definitely that, for civil engineering trusses of structural steel, the critical compressive unit stresses are practically at the yield point. Therefore it is evident that, for these trusses, the necessary stability analysis for the critical load is futile. This does not apply to the airplane truss (as in Table 5(a)), because the critical compressive unit stresses are considerably below the proportional limit. From a plot of column curves for structural carbon steel, for both pinned and fixed restraints,[14,18] it is readily observed that (in the usual range of L/r-ratios for civil engineering sections) the critical unit stress is close to the yield point. The relation between the buckling unit stress of a unit structural steel column (of the usual stocky section in civil engineering structures) and the yield point of the material has been discussed for many years. The suggestion has been made that the buckling unit stress be approximated at nine tenths of the yield point for columns in civil engineering structures. It is also known that a column, as part of a framework, is subject to conditions different from those in the member acting as a unit.

Prior investigators, particularly in the field of aeronautic structures, have indicated that, when a steel frame is being loaded, its compression members will drain possible reserve strength from the adjacent members. This fact is mentioned by the authors and has been independently verified by the writer.[14] Since previous research reveals that, for a unit column of the civil engineering type, the buckling unit stress is close to the yield point, by intuition it follows that the buckling unit stress of a column in a framework must be practically at the yield point because of the stress contributed by the possible reserve strength in the adjacent members. In fact, in discussions with a number of structural analysts the same general opinion was expressed to the writer although it was

not based on stability calculations. However, this relation does not apply to the airplane truss example, used by the authors and the writer, and it should be noted that stability analysis is required for such structures. The critical load factor for the airplane truss cited is 1.0309 in the paper and 1.0310 according to the writer—indicating a 3% safety factor on the basis of design load. It is evident therefore that a stability analysis is required for the airplane truss.

The authors state that it is advisable to use small simplified member groups of the truss instead of the entire framework to determine the critical load, and that the illustrative example is from an arrangement proposed by the writer.[14] The grouping for the airplane truss is shown in Fig. 10. Because of the limited size of the truss all the members are employed when the terminals are twice removed from the joint where the unit moment is applied. For simplicity in calculation fixed ends are used throughout. The grouping in Fig. 10 was recommended to Mr. Kavanagh because it was found applicable to both aeronautical and civil engineering trusses. The calculations for r are simple, and satisfactory values for k result in from close to the true values of the critical load. Based on the generalized application of the Saint Venant theory, it is evident that, the farther the terminals are from the joint where the unit moment is applied, the closer the results will come to the true answer. Furthermore, the use of a grouping with terminals twice removed from both ends of a compression member requires considerable calculations with different r-values at each end, up to the value of the equivalence r. In this case, there will be approximately the same variation from the true critical load as for the grouping in Fig. 10. With terminals twice removed, contribution to the stability limit is allowed from the twice removed tension members and from any possible low-stressed compression members. For the grouping in Fig. 10, the operation is applicable only to the members that enter the joint where the unit moment is applied, because no moment is carried back from the twice removed terminals to the joints once removed. Denoting the stiffness ratio of the members as S_r, instead of as $K/\Sigma K$; and, with C as the carry-over factor, the unbalanced moment r for a unit moment at joint U_1 in Fig. 10 is equal to

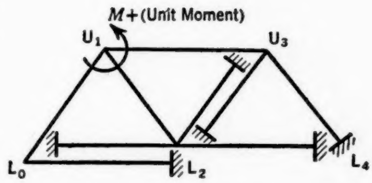

FIG. 10.—MEMBER GROUP, AIRPLANE TRUSS

$$r = \Sigma \begin{bmatrix} (S_r\,\overline{U_1L_0})\ (S_r\,\overline{L_0U_1})\ C^2 \\ (S_r\,\overline{U_1U_3})\ (S_r\,\overline{U_3U_1})\ C^2 \\ (S_r\,\overline{U_1L_2})\ (S_r\,\overline{L_2U_1})\ C^2 \end{bmatrix} \qquad (25)$$

Thus, a simple calculation instead of a lengthy moment distribution is available for satisfactory values of r.

The developments by the authors, and this discussion, are both based on planar trusses, with buckling within the plane, assuming the effect of joint translation to be negligible.

J. EDMUND FITZGERALD,[39] JUN. ASCE.—An original contribution to, and a survey of, the various methods of determining truss capacities are presented in this paper. It is to be regretted, however, that the authors did not include an example of an actual steel truss, preferably indeterminant, in order that the reader might be enabled to judge their method as applied to commonly encountered design problems.

The paper derives its value from the authors' presentation of a method of handling trusses with rigid joints whose $\frac{L}{i}$-values are greater than those normally encountered in practice, since for the usual truss the authors state (third paragraph following theorem 3):

> "* * * investigation has demonstrated clearly that the buckling loads for steel building and bridge trusses * * * are so close to * * * the yield stress * * * that there is absolutely no need for buckling load analyses."

The writer believes that the worth of the paper lies in the mathematical confirmation of the theory of limit design as originally presented by J. A. Van den Broek,[40] M. ASCE.

As will be shown, the writer disagrees with the limitation set by the authors and believes that there is no need of a buckling analysis for trusses with a slenderness ratio greater than those commonly encountered in practice. As a case in question, consider the truss shown in the example in Fig. 5. This truss is unquestionably in the slender range since the values of L/i for its members vary between $L/i = 140$ and $L/i = 170$.

In spite of the mathematically correct statement by the authors that "* * * when a truss buckles all members tend to fail simultaneously" (which would be difficult indeed to demonstrate in the field or in a laboratory), the writer will employ dictum (3), from Mr. Van den Broek's paper to solve the truss in Fig. 5.

In dictum (3) it is stated:

> "When in a * * * n-fold redundant structure, n redundants are stressed to * * * their critical buckling load, the deformations involved are of the order of magnitude of elastic deformations until an $(n + 1)$th member has reached * * * critical buckling capacity."

In order to provide for the redundants that are produced by the moment restraints at the various joints, the analyst may use the common design practice of assuming the effective lengths of the members as $L' = 0.75\,L$—that is, $K = 0.75$. For members with $\frac{L}{i}$-ratios between 100 and 200, this assumption is borne out quite well. For $\frac{L}{i}$-ratios less than 100, using the yield stress will give excellent results. This is in agreement with the authors' recomendations. For L/i greater than 200, the structure will probably be poorly designed. Thus, the assumption of 0 redundants is incorporated in the effective length and the designer may proceed to the analysis by limit design.

Since the truss shown has 0 redundants, $n + 1 = 1$, and the problem resolves itself into finding the weakest member—that is, in the authors' terms,

[39] Asst. Prof. of Civ. Eng., North Dakota State College, Fargo, N. Dak.

[40] "Theory of Limit Design," by J. A. Van den Broek, *Transactions*, ASCE, Vol. 105, 1940, p. 638.

finding the member which approaches failure the fastest (thus decreasing the reserve of strength in the structure to zero) and produces incipient failure.

Considering the truss as loaded with a load P at joint E and with loads 0.1 P at joints B and C the individual bar stresses are determined by statics. The entire calculation is shown in Table 7.

The values for the ultimate or buckling unit load were taken by interpolation from data published by Mr. Van den Broek. For $\frac{L}{i}$-values of 100 or so, the $\frac{L'}{i}$-ratio will give an ultimate unit load approximately equal to the yield point value.[41] The areas of the members and other pertinent data can be taken from the original Van den Broek paper.

The authors' overload factor of 1.03 represents a 1% difference from the result in Table 7. Actually, the variation of the physical and geometric constants of the truss itself is likely to exceed any error introduced by choosing a L'-value that is in error. The authors have found that the correct L'-value at failure is about 0.702. This represents a difference of almost 7%.

TABLE 7.—ANALYSIS OF THE TRUSS IN FIG. 5

Member	Stress	$\frac{L}{i}$	$\frac{L'}{i}$	Ultimate strength	Load	P	Overload factor
AB,CD.........	$-0.750\,P$	140	105	24,000	3,200	4,270	1.017 governs
BE,EC.........	$+0.625\,P$	140	....	36,000	4,800	7,740	1.840
BC............	$-0.875\,P$	170	127	19,000	3,520	5,620	1.340
AE, ED........	$+0.450\,P$	167	....	36,000	4,800	10,670	2.540

The tentative conclusion to be drawn from all this is that the theory of limit design may be used for the analysis and design of rigid trusses of any reasonable degree of slenderness ratios with the ensuing results in close agreement with those obtained by the more laborious, but mathematically more rigorous, method of this paper. From the designer's point of view, the paper should serve as another confirmation of the validity and universality of the theory of limit design. It should be mentioned that the authors' use of the ultimate load for design purposes rather than the allowable stress concept shows again how rapidly this newer and infinitely more logical procedure is taking hold. Let it be hoped that, in time, the building codes will recognize it at least as an alternate method of design.

The statement (second paragraph beyond Eq. 2)—

> "* * * the abutting members, instead of creating an end moment on the compression member as it starts to buckle, may have moments presented to them by the buckling member"

—seems to state the same thing twice—action and reaction, etc. Actually, the statement could be clarified to explain that the abutting members, instead of impressing a moment on the compression members of one sense before buckling

[41] "Theory of Limit Design," by J. A. Van den Broek, John Wiley & Sons, Inc., New York, N. Y., 1948, p. 85, Fig. 51.

occurs, which tends to buckle the column, will impress a moment of opposite sense on the column after it buckles, which tends to reduce the degree of buckling. This is the same phenomenon which occurs as reversed eccentricity when testing a flat-ended eccentric column.

In the "Synopsis," the authors make a statement which seems to persist in any discussion about columns. It represents a confusion of physical effect that even the Column Research Council has been guilty of on past occasions. To state that "* * * end restraints decrease and that the effective lengths increase as loads are increased * * *" is simply to state the same physical effect in two ways. The stiffness of the column may be determined by using a fixed value of end restraint, say, $S = 0$, and considering the effective length, L', to vary; or the effective length may be assumed constant, say, $L' = L$, and the degree of end restraint may be considered to vary. The stiffness effect in the writer's example was provided for by assuming $S = 0$ and $K = 0.75$. This was done to use the results of tests on pinned-end columns in the selection of buckling unit loads. It certainly represents a needless complication to consider both the variation in K and S since they are directly related;[42] thus:

$$K = L' = \frac{2}{3}\left(\frac{S + 3\, I/L}{S + 2\, I/L}\right) \qquad (26)$$

The writer should like to congratulate the authors on their clarity of presentation, and hopes that in the closing discussion there will be an example of the type mentioned at the beginning of this discussion.

JOSEPH S. NEWELL.[43]—The authors have summarized, clearly and concisely, the state of the art of stability determination as it is currently applied to trusses. Their excellent paper leads the reader to the conclusion that stability analyses of frames or trusses involve computations which, although not difficult, are tedious. It certainly indicates that further research is desirable to develop simpler methods of evaluating stiffness coefficients for various members or various joints. Table 2, which is used in determining such coefficients for trial loads 1.0119 times those used in the basic example,[13] is by no means short, yet it omits thirteen of the nineteen cycles actually made. Even so, it yields little definite information beyond the fact that the criterion shows the structure under investigation to be stable at this load and, hence, that the computations must be repeated for at least one increase in load if the limit of stability is to be determined exactly.

It is no reflection on the merit of the paper that so much work produces so little actual information. It is, rather, an indication of the present state of the art, an indication that great rewards may be had from studies that will enable analysts to approximate critical loads or stiffness coefficients by less tedious methods.

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[42] "Interrelation of Certain Structural Concepts," by Camillo Weiss, *Transactions*, ASCE, Vol. 111, 1946, p. 391.

[43] Prof. of Aeronautical Structural Eng., Massachusetts Inst. of Technology, Cambridge, Mass.

[13] "Airplane Structures," by A. S. Niles and J. S. Newell, John Wiley & Sons, Inc., New York, N. Y., 3d Ed., Vol. II, 1943

Perhaps the conclusion—that for compression members in steel trusses with the usual range of slenderness ratios the buckling stress coincides practically with the yield point of the steel regardless of end restraints—will eliminate the need for stability studies for many bridge and building structures. It will not do so in aircraft design, however, or in any structure that includes members subject to local buckling at stresses below the yield point of the material. These constitute an important fraction of the structures subject to stability investigations.

What should be regarded as the yield point? J. A. Van den Broek,[44] M. ASCE, has shown test results which indicate the peak of the curve before the "drop-of-the-beam" value to be the desired yield. Many engineers may accept this value, or the drop-of-the-beam value for mild steels, but what is to be used for alloy steels or aluminum alloys whose yield stresses are established arbitrarily?

Theorems 1, 2, and 3 cannot be stated with sufficient emphasis. They present points which many engineers have never considered, and few have analyzed to the extent that they clearly understand the action of a truss when it reaches a condition of instability. Whether the "series" or the "stiffness" criterion be used, it is important to realize that in a truss which has become unstable the reserve strength and stiffness in every member and joint have been exhausted, the unstable members undergo deflections, and the unstable joints undergo rotations, which are not proportional to the loads on the structure. The authors are to be congratulated on the clarity of their discussion of these points.

Now that they have summarized the long and tedious procedures in current use, it is hoped that the authors, or others inspired by their work, can evolve approximate methods which, although they may not eliminate a final "series-criterion" analysis, will reduce the preliminary or trial-and-error computations through which the critical load is estimated. The current procedure is not difficult to understand, but it is far too tedious to be practicable for routine application to structures having a large number of members.

Charles W. Dohn,[45] Jun. ASCE.—It is not entirely clear to the writer as to what constitutes "primary instability" of trusses and he would appreciate a "word picture" illustration or further explanatory examples. When a truss fails by primary instability do the individual members buckle and tear, and why should this be any different from what is commonly defined by the terms "ultimate load" and "final failure?" Are primary instability and critical load to be thought of as precise terms defining or predicting the exact loading point at which the compression members of a truss will buckle (taking into account relaxation in end restraint with increase in load); and are these terms applicable only where the effect of joint translation is small?

Compression members for trusses are commonly designed more or less as free-ended columns according to empirical formulas. This supposedly proportions them to carry a definite load with reasonably known safety. When a

[44] "Euler's Classic Paper 'On the Strength of Columns,'" by J. A. Van den Broek, *American Journal of Physics*, July–August, 1947, p. 309.

[45] Engr., New York, N. Y.

truss fails as a result of overload, the exact point and method of failure ordinarily would be impossible to predict, in most cases. However it seems safe to assume that approximately the same factor of safety that applied to the design of the individual members would apply to the truss as a whole. The authors' investigations verify this assumption by a wide margin for trusses in which the effect of joint translation is small and compression members buckle as their end restraints become less and less with increasing load. The authors state that they have found that the limiting load coincides practically with the yield point of the steel regardless of end restraints. Cannot truss compression members be thus designed for a uniform allowable stress of the same value as that used for tension members, plus a minimum l/r, and thereby dispense with column formulas for truss compression member design? It should be borne in mind that compression members cannot be reasonably designed to have the same ultimate strength as tension members in any case. Such methods of design might be limited to trusses with small joint translations. How may designers simply determine when a truss has joint translations small enough to include such a method of design? Could a ratio of truss deflection divided by truss span or truss depth over truss span, or some similar ratio, be set up to define such limits, and what would be some values for such a ratio for small, medium, large, and excessive joint translations? Could such methods of design be practicable?

In discussing theorem 3, the authors state: "* * * the method is applicable only to trusses, where the effect of joint translation is small"; and, in discussing theorem 2: "* * * secondary or localized failure is not being discussed. Actually, such failure would generally occur before the truss failed as a whole." It would appear that joint translation and secondary stresses are likely to be effective factors in a truss with rigid connections and reserve "elements of strength associated with bending resistance." Failure of connections would generally precede buckling of compression members. This can be deduced by logic. Before a compression member will fail by buckling, it must deflect more than the full elastic range. In approaching this condition the joint displacements will be such that the elastic range of the joint material will be exceeded and the joint will yield. The joint thus functions in a somewhat unpredictable manner, following the magnitude and direction of the secondary stresses, and in the manner of framing compression members. Although it might be argued that such yielding relieves the secondary stresses, nevertheless, it places the truss one step closer to complete failure inasmuch as the restraining effects of the joints might be changed and the compression members are likely to be more easily subject to failure by buckling. Removal of the restraining effects of joints by secondary stress failure will involve joint rotation; and, since the direction which the buckle assumes is immaterial, it will naturally form in the direction of the greatest rotation caused by the secondary stress failure. Thus, the Euler formula factor, k, might even be assumed to have a value greater than 2—if indeed this factor might still be considered applicable. The factor of safety of the truss as a whole may, therefore, be somewhat less than the factor of safety for individual members in the case of a truss with large joint translations. This suggests that trusses might be grouped into three classifications: (1) Trusses with small joint translations which may be designed for

uniform allowable stress; (2) trusses with medium joint translations which may be designed with the aid of ordinary column formulas; and (3) trusses with large joint translations which may be designed as in class (2), but with an increased safety factor varying with joint translation. On the other hand, all compression members of a truss might be designed based on a given allowable stress with a deducting factor to vary directly with joint translation or truss deflection divided by truss span, or truss depth over truss span, including perhaps a factor based on maximum, minimum, or average lengths of members. A fairly low minimum slenderness ratio would be specified.

As a result of further considerations it might be possible to vary stability and obtain a wider range of elastic action preceding failure up to the limits defined by primary instability by using different materials. For instance, joint material might be made of aluminum, with compression members of steel. Such joints might be simple aluminum blocks of standard dimension with an insulative cover, and with holes for bolts or rivets. In an aluminum truss, short steel cover plates might be added to compression members at points of critical buckling. Trusses so built and designed might serve to be doubly useful as stiffening trusses for suspension bridges where the dampening value of a truss built of materials with different physical properties can be utilized to counteract aerodynamic instability.

Can continuous trusses and a continuous tied arch truss, such as those used in the Dubuque (Iowa) crossing[46] of the Mississippi River, be properly considered in the category of steel trusses with the usual range of slenderness ratios? Could not rotation resulting from secondary stress failure be added to changes in end restraint, with increase in load to cause buckling of extremely long members carrying heavy loads? Such buckling might occur at the supports of continuous trusses, particularly in trusses that deflect considerably.

The writer would be most happy to go on designing usual trusses with usual slenderness ratios and usual allowances for secondary stress. He is most grateful to the authors for giving pertinent factors concerning compression members and buckling loads as related to end restraints on truss members. He feels that most designers might find primary instability of academic interest but difficult to study and to apply to actual truss design problems unless the concept leads to definite recommendations in the form of design specifications.

Harold E. Wessman,[47] M. ASCE, and Thomas C. Kavanagh,[48] Assoc. M. ASCE.—The contributions of the discussers are appreciated. The writers are particularly indebted to Mr. Newell for focusing attention on theorems 1, 2, and 3. From a philosophical viewpoint, these theorems are extremely important. Mr. Winter, on the other hand, overlooked this and devoted his primary attention to advancing the argument that the end restraint method developed by P. T. Hsu[17] is superior to the extension of the Lundquist-Hoff method presented in the paper. The writers do not accept this argument, and will refute it presently; but, lest the general reader be led astray by a discussion

[46] Discussion by Charles W. Dohn of "Mississippi River Bridge at Dubuque, Iowa," by R. N. Bergendoff and Josef Sorkin, *Transactions*, ASCE, Vol. 114, 1949, p. 1299.

[47] Dean, College of Eng., Univ. of Washington, Seattle, Wash.

[48] Prof. of Civ. Eng., Pennsylvania State College, State College, Pa.

of techniques, the writers will state at this point their frank opinion that there is no advantage in using either one of the methods as presented. Quoting from Mr. Newell's discussion, both are: "* * * far too tedious to be practicable for routine application to structures having a large number of members."

The paper high lights the need for a simple approximate method for determining the stability limit for trusses. Mr. Slavin has presented one method of simplifying the solution. It involves the use of groups of members into which the truss may be resolved. These groups, in which external ends of boundary members twice removed from a reference joint are fixed, are quite satisfactory for determining the buckling load within close limits. However, this approximate solution is not enough of a simplification for design office practice. Further simplifications, such as those presented by one of the writers (Mr. Kavanagh) in his research thesis[18] presented to New York University, in New York, N. Y., offer the most promise. The most important of these is the procedure which determines the critical loading from one trial load, which the writers feel goes a long way toward meeting the objectives set up by Mr. Newell. This procedure will be outlined briefly.

Prior to any actual analysis, the limiting range of the buckling load should be established. The following theorem[18] will prove helpful:

> Theorem 4. For a rigid-jointed truss framework, in the range of loading between that multiple of load which first produces $L/j = \pi$ in any compression member, and that which first produces $L/j = 2\pi$ in any compression member, there can be at most one critical load.

Accordingly, once the range of loading limits is known (and it is easily computed), the trial-and-error procedure which characterizes the stability analysis can begin at any loading in this range without fear of encountering higher buckling loads or missing the lowest buckling load. (The exception to this theorem is the pin-ended strut, which has a buckling mode at $L/j = \pi$ and another at $L/j = 2\pi$, but this case is so easily recognized that it does not impair the general applicability of theorem 4.) This range of possible limits is often so very narrow as to make any further formal analysis unnecessary.

Of equal importance, the preliminary survey undertaken to set the range of buckling values gives, at once, an indication of the "weakest member" as well as of the "weakest joint." It also identifies those compression members which require restraint (so-called "buckling" members) and those compression members which actually have a reserve of strength to offer to "weaker" members. At the end of his formal analysis, Mr. Winter discusses facts which most analysts recognize in the basic procedure of setting up any problem.

An analysis of a great number of different types of practical trusses has indicated that satisfactory approximations can be attained by considering groups of members, rather than the truss as a whole. From a practical point of view, however, the time required to analyze a truss, by groups extending more than one joint from any reference member or joint, is usually excessive. Of all the group patterns studied, that group based on fixed (or pinned) joints once removed from the ends of each "buckling" compression member has been found to give the most satisfactory approximation, and at the same time to offer a close approach to the physical action of a restrained member, which

would be desired from a design viewpoint. The use of fixed terminals is not essential to this procedure, but it has been found entirely satisfactory, even though slightly on the unsafe side, and has the added advantage of eliminating the need for one set of tabular stiffnesses (S''). Pin ends can be employed if desired.

A noticeable disadvantage of this grouping lies in the fact that all adjacent members are considered as restraining members, even though some might themselves be distressed compressive members (buckling members) incapable of offering restraint. An improvement in accuracy in all cases is afforded by a modification of the grouping by more closely estimating the true restraints. One selects each buckling compression member ($L/j > \pi$) and considers it restrained at each end by a proportion of the stiffness of all adjacent members (far ends fixed), excluding other adjacent buckling members. The total stiffness of the restraints is thus divided sensibly among those distressed members actually requiring support, in line with the true physical action of the structure. As an example, the airplane truss would be divided into groups centering on the buckling members AB, BC, and CD, as shown in Fig. 11.

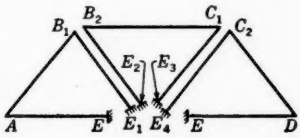

FIG. 11

Although the proportionment (for example, of the stiffness of member BE between buckling members AB and BC) by a formula may be fairly accurate, it has been found that any reasonable proportionment may be assumed. The error in the proportionment will be evident from the presence of different r-values when a unit moment is applied at joint B in the group AB_1E_1, and another at joint B in the group $E_2B_2C_1E_3$. This error may be readily corrected, for it has been observed that the geometric mean of all the r-values at a joint is usually close to the true r-value for the joint in the truss as a whole. This observation reaffirms an observation made by the French investigator, Henri Rivière,[49] who used the geometric mean successfully in a stability problem of slightly different character.

The importance of the foregoing method of grouping lies in the fact that the restraints on any buckling compressive member for a trial loading in the range of possible values determined by theorem 4 are primarily tension members whose stiffness changes very slowly and which may be considered constant within the range of analysis. Accordingly, it is immediately possible to give an approximate value of the true buckling load of the truss, once the results (or even some preliminary data) are available from one trial load. It is immaterial which technique is employed to jump from the trial load to the estimated critical load; the transition can be accomplished by the Zimmermann formula (Eq. 13*b*), or by a modification of the Lundquist procedure, or by other methods. The important fact is that the particular grouping allows for a substantially constant set of restraints on each compression member within the range of analysis defined by theorem 4, and for the determination of an approximate critical load within 1% or 2% of the true value.

[49] "Calcul des Poutres Comprimées Encastrées Elastiquement à leurs Extremités," by Henri Rivière, *L'Aerotechnik* (supplement to *L'Aeronautique*), No. 200, January, 1936, pp. 7–12.

For example, applying the data for trial load factor 1.0119 to the grouping of Fig. 11, and arbitrarily dividing the stiffness of member BE equally between members AB and BC, one obtains the calculation in Table 8 from simply using the Zimmermann formula (Eq. 13*b*) or the chart of Fig. 4. The inequality of the load factors in Table 8 indicates the error in the assumed proportionment of the stiffness of member BE. Obviously, member BC requires more restraint than member AB, and a more judicious proportionment might have been assumed, particularly since it is known in advance that BC is the weakest member.

The estimated critical load factor is $\sqrt{1.145 \times 0.923} = 1.028$—only 0.3% less than the true value.

Up to this date, the method requiring the least number of trial loads was the Lundquist adaptation of the Southwell procedure[15] for determining critical

TABLE 8.—ANALYSIS OF FIG. 11 USING DATA FOR ONE LOAD FACTOR (1.0119)

Member (Fig. 11)	Total stiffness, s	$\frac{E' I}{4 L}$	Restraint factor, k	Critical load, P_{cr} (lb)	Load factor	Geometric mean
AB—						
s_A	13,519.1	0.1903	0.685	3,607	1.145	1.028
s_B	7,971.8	0.3226				
BC—						
s_B, s_C	7,971.8	0.4061	0.745	3,198	0.923	

loads in columns from test data. This technique required a minimum of three trial loads and, in all problems worked by the writers, proved far too time consuming.

Previously in this closing discussion, the writers have emphasized the philosophical significance of theorems 1, 2, and 3. From a practical viewpoint, however, the most important conclusion is presented in the statement in the paper (three paragraphs beyond theorem 3) that:

> "* * * additional investigation has demonstrated clearly that the buckling loads for steel building and bridge trusses, with members having slenderness ratios in accord with current design practice, are so close to the loads corresponding to the yield stress of the steel employed, that there is absolutely no need for buckling load analyses."

In this connection, Mr. Winter states that: "Some skepticism concerning this contention seems justified." To illustrate his point, he uses a roof truss with slenderness ratios varying from 145 to 175. Such a truss is decidedly not typical of current design practice. The main compression chords in most bridge trusses have slenderness ratios under 75. The slenderness ratios of main compression members in roof trusses may be a little higher, but it is most unusual to find chords in roof trusses with slenderness ratios greater than 100.

[15] "A Method for Estimating the Critical Buckling Load for Structural Members," by Eugene E. Lundquist, *Technical Note No. 717*, National Advisory Committee for Aeronautics, Washington, D. C., July, 1939.

Compression diagonals near the centers of trusses often have slenderness ratios in the range of from 100 to 120, but these diagonals seldom dominate the stability of the truss. In most cases, their working stresses for full-span loadings are relatively low. They are governed by part-span live loads which do not result in maximum stresses in adjacent chord members.

Fig. 12 demonstrates why the buckling load is so close to the yield strength for most steel members. This figure gives the theoretical buckling curves for

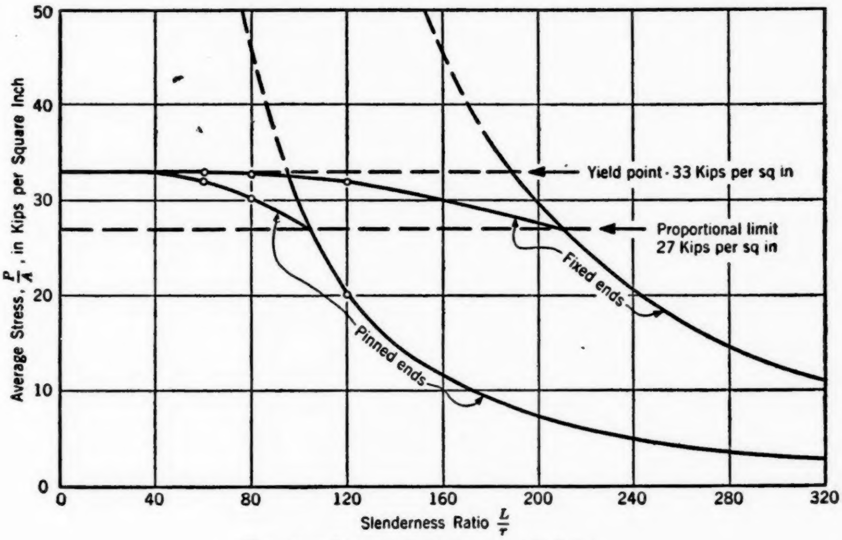

FIG. 12.—COLUMN CURVES FOR A-7 STEEL

ideal straight columns of an A-7 steel, plotted from the Euler formula (Eq. 2) in which E', the effective modulus, is the tangent modulus:

$$\frac{P}{A} = \frac{\pi^2 E'}{(k\ L/r)^2} \qquad (27)$$

The tangent modulus values are shown in the curves in Fig. 13 for a stress-strain diagram which is based on assumed values for the proportional limit of 27 kips per sq in., at a strain of 0.0009 in. per in., and a yield point of 33 kips per sq in. (the minimum specified for A-7 steel) at a strain of 0.0013 in. per in., with a parabolic variation between these points. The modulus shown in Fig. 13 is the tangent modulus.

One of the curves (Fig. 12) is plotted for a column with pinned ends; the other, for a column with fixed ends. Obviously, for a column with an intermediate degree of restraint, the column curve will be represented by a line lying somewhere between the two limiting curves shown. For slenderness ratios less than 100, all three curves are close to one another and also close to the yield point value. This illustrates graphically that the effect of end restraints on buckling loads of columns in steel building and bridge trusses, with slenderness ratios in accord with usual design practice, is small. Furthermore,

the stresses corresponding to the buckling loads in the lower slenderness-ratio range are so close to the yield point that the latter may be considered as a practical ceiling value for design within this range (see Table 9).

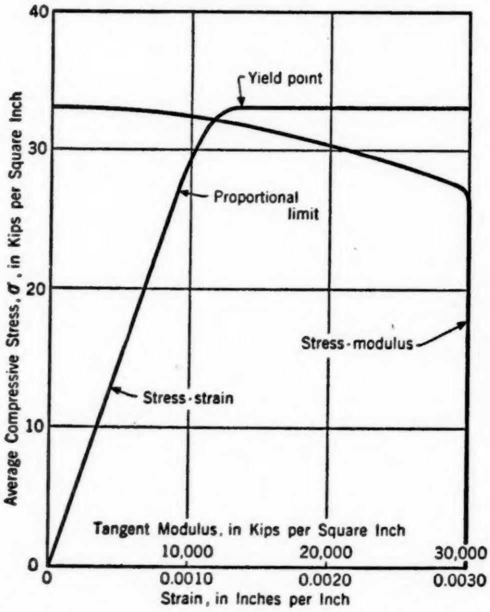

FIG. 13.—STRESS-STRAIN AND STRESS-MODULUS CURVES FOR A STRUCTURAL CARBON STEEL (YIELD POINT, 33 KIPS PER SQ IN.)

For a slenderness ratio of 75, Table 9, the stress corresponding to a critical load for an average condition of end restraint is 97% of the yield point stress.

It is true that the yield point value is slightly on the unsafe side. On the other hand, although the specified yield point for A-7 steel is 33,000 lb per sq in., the actual yield point in many steel samples generally exceeds 33,000. Working stresses are based on the minimum value.

The writers believe that it is academic to make involved stability analyses for steel trusses of structural carbon steel. The same conclusion holds for most silicon steel and low alloy steel trusses. However, as Mr. Newell notes in his discussion, this conclusion is not applicable for the type of trusses encountered in aircraft design using aluminum alloys whose yield stresses are established arbitrarily.

Mr. Benjamin states that: "Two factors are neglected by the authors—influence of secondary stress moments and moments caused by initial curvature." The discussion does not indicate clearly what is meant by the "influence of secondary stress moments." The moments associated with end restraints are those which are created by rigid connections. They are the moments that cause secondary stresses. It is doubtful if secondary moments are of much significance in members with such a large slenderness ratio that failure would be caused by primary instability rather than local yielding due to localized stress intensity. John I. Parcel, M. ASCE, and Eldred B. Murer,[50]

TABLE 9.—CRITICAL LOAD RATIOS

Stress ratios	Slenderness Ratios			
	50	75	100	125
$\frac{P_{cr} \text{ (for fixed ends)}}{P_{cr} \text{ (for pinned ends)}}$	1.01	1.06	1.17	1.67
$\frac{\text{Average } P_{cr}/A}{P/A \text{ at yield point}}$	0.99	0.97	0.91	0.77

[50] "Effect of Secondary Stresses Upon Ultimate Strength," by John I. Parcel and Eldred B. Murer, *Transactions*, ASCE, Vol. 101, 1936, p. 289.

Assoc. M. ASCE, indicate that: "* * * the ultimate strength is practically unaffected, even by high secondary stresses * * *."

Mr. Fitzgerald notes his belief that the worth of the paper lies in the mathematical confirmation of the theory of limit design. As noted by the writers (first paragraph following Eq. 16): "* * * the phenomenon of truss buckling is a beautiful illustration to support the theory of limit design." Mr. Fitzgerald furthermore presents the view (which is quite opposite from that presented by Mr. Winter) that there is no need of a buckling analysis for trusses with a slenderness ratio greater than those commonly encountered in practice. Mr. Fitzgerald's conclusion evidently was based on the fact that the usual design assumption of an average value of k (namely, 0.75) was not far from the true k-values obtained in the analysis. The writers wish to emphasize, however, that this would not have been true had the tension members been designed, say, as flat bars capable of offering little flexural restraint.

Concerning the arguments advanced by Mr. Winter in favor of the Hsu method of calculating end restraints by iteration, as compared with the procedure using moment distribution, it can be shown in general[18] that any method of indeterminate analysis may be employed for stability analysis, provided the effect of axial loads is introduced. The moment distribution concepts, already widely known and used, require no new theory, no modification of physical ideas, no special graphs, formulas, or charts or other aids for their use in stability analyses. (Tables of modified stiffness and carry-over factors are now standard tools for aeronautical engineers in beam-column work, and even civil engineering texts are beginning to include these data.) On the other hand, many methods of indeterminate analysis based on "fixation factors," "degree of fixity," "degree of restraint," etc., have been advanced in the past; [9,51,52,53,54,55,56] yet it is doubtful if these can be said to have gained widespread usage.

The compactness, neatness, and, above all, the self-contained nature of the moment distribution process are factors which, in the writers' opinion, make this seeming "detour" a far smoother road to travel. Once the moment distribution has been set up (and all methods require about the same amount of preliminary setup work), the computation flows by simple arithmetical methods to a solution, without distraction from auxiliary equations or graphical aids. The fact that the entire method centers around a single all-inclusive factor, r, upon which one focuses his entire attention during the course of the analysis, makes this technique a far more unified and powerful tool.

Although the two methods under discussion appear to employ different techniques, they are fundamentally very closely related, and any investigator faced with the need for extended analyses would be amply repaid by a study of

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[9] "Knickfestigkeit der Stabverbindungen," by H. Zimmermann, W. Ernst & Sohn, Berlin, 1925.

[51] "Statically Indeterminate Frameworks," by Thomas F. Hickerson, Univ. of North Carolina Press, Chapel Hill, N. C., 3d Ed., 1949.

[52] "Analysis of Continuous Frames by the Method of Restraining Stiffnesses," by Earle B. Russell, Ellison and Russell, San Francisco, Calif., 1934.

[53] "Continuous Beam Structures; a Degree of Fixity Method and the Method of Moment Distribution," by Eric Shepley, Concrete Publications, Ltd., London, 1942.

[54] "Rigid Frames," by L. T. Evans, Edwards Bros., Ann Arbor, Mich., 1938.

[55] "Neue Statik der Tragwerke aus Biegesteifen Stäben," by Max Mayer, Bauwelt, Berlin, 1937.

[56] "Allgemeine Theorie des Elastisch Eingespannten Balkens," by M. Ritter, *Publications*, International Assn. for Bridge and Structural Eng., Vol. II, 1933, p. 290.

the method which Mr. Winter and his associates have so thoroughly explored. It is not correct to conclude that a moment distribution solution for one trial loading gives only the lower limit of the critical load. Aside from the fact that the need for bracketing the critical load is eliminated by the approximation procedure previously described (which yields the critical load immediately), bracketing can be accomplished by moment distribution in a number of ways, thus—

(*a*) Apply a unit moment at the end of the "weakest" member (usually the "weakest" joint), and from the *r*-values, determine restraints and upper and lower limits of P_{cr} by a procedure analogous to that indicated by Mr. Winter; or

(*b*) Using a purely moment distribution approach, "force" the total stiffness ($\Sigma S'$) of the weakest joint to zero by allowing only the "weakest" member to vary. Thus, in the airplane truss, using the data previously calculated for $\Sigma S'_b$, one may write immediately: $S_{bc} - 0.7192\ (17{,}185.6 + S_{bc}) = -12{,}163.3$. Consequently: $S_{bc} = 665$; $\frac{S\,L}{E'\,I} = 0.0513$; and $L/j = 4.562$ for member BC.

This procedure gives the upper bound of the critical load factor as 1.10. As stated, however, these bracketing procedures may be dispensed with if the direct approximation procedure is utilized.

Mr. Dohn states that:

> "* * * most designers might find primary instability of academic interest but difficult to study and to apply to actual truss design problems unless the concept leads to definite recommendations in the form of design specifications."

It is hoped that the studies made by the writers, and those which are being made elsewhere under the egis of the Column Research Council, will eventually lead to definite recommendations which will be incorporated in design specifications and which will reflect, more scientifically, a knowledge of the behavior of compression members in the different types of structures in which they are used.

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